Exercise sheet 3

Exercise 1. Show that the Dirac mass $\delta_0 : C_c(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$\delta_0(\varphi) = \varphi(0)$$

is a continuous and linear functional on $C_c(\mathbb{R}^n)$ which can be extended, thanks to Hahn-Banach theorem, to a continuous and linear functional on $L^\infty(\mathbb{R}^n)$. Moreover, show that there does not exist any function $f \in L^1(\mathbb{R}^n)$ such that

$$\delta_0(g) = \int_{\mathbb{R}^n} fg\,dx \quad \text{for all } g \in L^\infty(\mathbb{R}^n).$$

Exercise 2. Show that the dipole functional $\delta_0' : \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ defined as

$$\delta_0'(\varphi) = -\varphi'(0)$$

is a distribution, i.e. an element of $\mathscr{D}'(\mathbb{R})$. Moreover, show that there exists no Radon measure μ such that

$$\delta_0'(\varphi) = \int_{\mathbb{R}} \varphi\,d\mu.$$

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be an open set and $(T_k)_k \subset \mathscr{D}'(\Omega)$ be a sequence of distributions which is weakly converging to $T \in \mathscr{D}'(\Omega)$. Show that for any multi-index α the sequence $(D^\alpha T_k)_k$ is weakly converging to $D^\alpha T$.

Exercise 4. For every $n \in \mathbb{N}$ consider the distributions in $\mathscr{D}'(\mathbb{R})$ given by

$$T_n = \delta_{\frac{1}{n}} \quad \text{and} \quad S_n = n(T_n - T_{2n}).$$

Compute the respective weak-$*$ limits of $\{T_n\}_{n\in\mathbb{N}^*}$ and $\{S_n\}_{n\in\mathbb{N}^*}$ in $\mathscr{D}'(\mathbb{R})$.

Exercise 5.

- Compute the distribution $e^x \cdot \delta_0''$.
- Given $a, b > 0$, compute the distributional derivative of

$$f_{a,b} = H(x)\log|a\,x| + H(-x)\log|b\,x|,$$

where $H(x)$ indicates the Heaviside function.